

Notes.

- (a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.
- (b) We use $\mathbb{R}$ = real numbers.
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1. [40 Points] Let $f(x, y) = y^2 - x^2(x + 1)$.

- (i) Find all $c \in \mathbb{R}$ for which the level surface $f^{-1}(c)$ is a nonempty plane curve.
- (ii) Find a nonzero smooth vector field $\mathbf{X}$ on $\mathbb{R}^2$ that is orthogonal everywhere to the gradient vector field ∇f .
- (iii) Find a suitable open set U in $\mathbb{R}^2$ and a smooth vector field $\mathbf{Y}$ on U such that $\alpha(t) = (t^2 - 1, t(t^2 - 1))$ defined on the interval $I = (-1, 1)$ is a maximal integral curve of $\mathbf{Y}$.
- (iv) Prove that $\alpha(t)$ from (iii) when defined on the interval $(-2, 2)$, cannot be the integral curve of any smooth vector field on any open subset of $\mathbb{R}^2$.

2. [15 Points] Define the Gauss map of an oriented n -surface S in $\mathbb{R}^{n+1}$. Fix a nonzero element a in $\mathbb{R}^{n+1}$. Show that the spherical image of S is contained in the n -plane orthogonal to a iff for every $p \in S$, there exists an open line segment $p + ta$ (for t in some open interval in $\mathbb{R}$ around 0) through p that lies in S .

Hint: Is the constant vector field $\mathbf{X}(q) = (q; a)$ tangential at every $p \in S$?

3. [15 Points]

- (i) Verify that for any nonzero $c \in \mathbb{R}$ and any pair of orthonormal unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^3$, the great circle $\alpha(t) = (\cos at)e_1 + (\sin at)e_2$ is a geodesic on the unit sphere S in $\mathbb{R}^3$.
- (ii) Give an example of a two piecewise smooth parametrised curves $\alpha(t), \beta(t)$ on the unit sphere S in $\mathbb{R}^3$ having the same starting point $p \in S$ and ending point $q \in S$ for which the parallel transport of a vector $\mathbf{v} \in S_p$ results in different tangent vectors at q .

4. [15 Points] Let p be a point on an n -surface S in $\mathbb{R}^{n+1}$. Define the covariant derivative $D_{\mathbf{v}}$ of any smooth tangent vector field $\mathbf{X}$ with respect to $\mathbf{v} \in S_p$. If $\mathbf{Y}$ is also a smooth tangent vector field, prove that $\nabla_{\mathbf{v}}(\mathbf{X} \cdot \mathbf{Y}) = (D_{\mathbf{v}}\mathbf{X}) \cdot \mathbf{Y}(p) + \mathbf{X}(p) \cdot (D_{\mathbf{v}}\mathbf{Y})$.

5. [15 Points] For a point p on an n -surface S in $\mathbb{R}^{n+1}$ oriented by a unit normal vector field $\mathbf{N}$, define the Weingarten map L_p . For any $\mathbf{v} \in S_p$ and any parametrised curve $\alpha(t)$ on S with $\alpha(t_0) = p, \dot{\alpha}(t_0) = \mathbf{v}$, prove that $L_p\mathbf{v} \cdot \mathbf{v} = \ddot{\alpha}(t_0) \cdot \mathbf{N}(p)$.